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Spline Approximation by Quasiinterpolants*

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1. INTRODUCTION

Let Ω be a region in R^n, π a rectangular partition of R^n, and $S_\pi{}^k$ ($k \geqslant 1$) the corresponding spline space of degree $k - 1$. In this paper we shall explicitly construct for each function $f \in C^k(\Omega)$[1] a spline $F_\pi f \in S_\pi{}^k$, which we call the *quasiinterpolant* of f, having the following properties:

(i) $F_\pi f$ is *local* in the sense that its value at a point x depends only on the values of f in a uniformly small neighborhood of x.

(ii) F_π reproduces polynomials; $F_\pi(x^\gamma) = x^\gamma$ for $|\gamma| < k$.

(iii) $F_\pi f - f = \mathcal{O}(|\pi|^k)$.

Moreover, our quasiinterpolant has the rather simple form

$$F_\pi f(x) = \sum_j \sum_{|\alpha| < k} \omega_{j,\alpha} D^\alpha f(\tau_j)\, N_{j,k}(x), \tag{1.1}$$

where $N_{j,k}$ is the n-dimensional B-spline, τ_j is an arbitrary point in the support of $N_{j,k}$, and the weights $\omega_{j,\alpha}$ are given by (2.6) for $n = 1$ and by (4.3) for $n > 1$.

The literature on direct constructions of spline approximations like (1.1)

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[1] Standard multiindex notation [1, p. 1] will be used in this paper.

appears to be the following. In [3] Birkhoff has defined a scheme of *spline approximation by moments* ($n = 1$) of the form

$$P_\pi f = p(x) + \sum \left\{ \int w_i(t) f^{(2m)}(t)\, dt \right\} G(x, x_i), \tag{1.2}$$

where p is a polynomial of degree $2m - 1$ and $G(x, y)$ is a suitable Green's function. In the Appendix, we shall show that (1.1) in one dimension and (1.2) are in fact equivalent, and, hence, our quasiinterpolant provides an algebraic simplification of Birkhoff's scheme in this case. Using the Fourier transform Babuska [2], Strang and Fix [8, 13] have constructed approximations analogous to (1.1) for uniform meshes. Finally, in [4] de Boor proved the existence of projectors like F_π in (1.1). This work was subsequently generalized to n dimensions by Schultz [12] through the use of tensor products.

We note that it is straightforward to extend our construction to Chebyshev splines.

2. The Quasiinterpolant

Let k be a positive integer. We say that $\pi = \{x_i\}_{i=-\infty}^{\infty}$ is a *k-extended partition* for the open (finite or infinite) interval $I = (a, b)$ provided

(i) $x_i \leq x_{i+1}$, all i,

(ii) $\lim_{i\to-\infty} x_i = a$, $\lim_{i\to\infty} x_i = b$,

(iii) if d_i is the frequency with which the number $x = x_i$ appears among the x_i's, then $d_i \leq k$, all i.

With π a k-extended partition for (a, b), let $C_\pi^{(k-1)}$ denote the linear space of all functions defined on (a, b) with the following properties:

(i) $f \in C^{(k-1)}(x_i, x_{i+1})$, all i;

(ii) for all i and all $r < k$, $f^{(r)}(x_i -)$, $f^{(r)}(x_i +)$ exist (and are finite); and

(iii) for all i, $f^{(r)}(x_i -) = f^{(r)}(x_i +)$, all $r < k - d_i$. The following agreements will be convenient: If the function f has a jump discontinuity at $x = \xi$, then $f(\xi)$ stands for either $f(\xi+)$ or $f(\xi-)$. Further, the statement,

$$f(\xi) = g(\xi),$$

stands for the two statements

$$f(\xi+) = g(\xi+) \quad \text{and} \quad f(\xi-) = g(\xi-).$$

We denote by $S_\pi^{\,k}$ the linear subspace of $C_\pi^{(k-1)}$ consisting of all polynomial splines on π of order k (or, degree $< k$). Specifically, $f \in C_\pi^{(k-1)}$ is in $S_\pi^{\,k}$ if and only if

$$f^{(k)}(x) = 0, \qquad \text{for all} \quad x \notin \pi.$$

According to Curry and Schoenberg [6], the set of normalized B-splines on π,

$$N_{i,k}(x) = (x_{i+k} - x_i)\, g_k(x_i, \ldots, x_{i+k}\,;\, x), \qquad \text{all } i, \tag{2.1}$$

with

$$g_k(s; x) = (s - x)_+^{k-1} = \begin{cases} (s - x)^{k-1}, & s > x \\ 0, & s < x \end{cases} \tag{2.2}$$

forms a basis for $S_\pi^{\,k}$ in the following sense. Every $f \in S_\pi^{\,k}$ can be written in the form

$$f = \sum_i a_i(f)\, N_{i,k}$$

for exactly one biinfinite sequence $(a_i(f))$ of coefficients. Here, the biinfinite sum is to be formed *pointwise*, i.e.,

$$\left(\sum_i a_i(f)\, N_{i,k}\right)(x) = \sum_i a_i(f)\, N_{i,k}(x), \qquad \text{all} \quad x \in (a, b). \tag{2.3}$$

The right side of (2.3) is well defined since no more than k of the $N_{i,k}$ are not zero at any particular x.

For $f \in C_\pi^{(k-1)}$, we define an approximation $F_\pi f$ to f in $S_\pi^{\,k}$ by

$$F_\pi f = \sum_j (\lambda_j f)\, N_{j,k}\,, \tag{2.4}$$

where, for each j, λ_j is the linear functional given by the rule

$$\lambda_j f = \sum_{r<k} \omega_{j,r} f^{(r)}(\tau_j), \tag{2.5}$$

with

$$\begin{aligned} \omega_{j,r} &= (-1)^{k-1-r}\, \psi_j^{(k-1-r)}(\tau_j)/(k-1)!, \qquad r < k, \\ \psi_j(x) &= (x_{j+1} - x) \cdots (x_{j+k-1} - x), \end{aligned} \tag{2.6}$$

and τ_j some point in (a, b). Should τ_j be one of the points of π, then τ_j in (2.5) is to be replaced by τ_j+ or by τ_j-. In the few cases where it matters which choice is taken, we will say so.

The motivation for this somewhat complicated definition is twofold. For one, F_π can be shown to reproduce polynomials, i.e., $F_\pi p = p$, for all polynomials p of degree $< k$. Also, if, in particular, $\tau_j \in (x_j\,, x_{j+k})$, all j, then

$F_\pi f$ is a local approximation to f in the sense that $F_\pi f$ on (x_i, x_{i+1}) depends only on the behavior of f on (x_{i+1-k}, x_{i+k}). This is due to the fact that $N_{j,k}$ has its support in (x_j, x_{j+k}), all j. These two facts are combined in Section 3 to show that, for $x \in (x_i, x_{i+1})$ and $f \in C^{(k-1)}$,

$$|(f - F_\pi f)(x)| \leqslant K(x_{i+k} - x_{i+1-k})^{k-1}\, \omega(f^{(k-1)}, x_{i+k} - x_{i+1-k}), \tag{2.7}$$

with K an absolute constant. In addition, one obtains corresponding estimates for the degree of approximation to $f^{(r)}$ by $(F_\pi f)^{(r)}$, $r < k$.

LEMMA 2.1. *Let λ_j be given by* (2.5) *and* (2.6). *If x is any point and p is a positive integer no bigger than k, then*

$$\lambda_j(\cdot - x)^{k-p} = \frac{(k-p)!}{(k-1)!}(-1)^{p-1}\,\psi_j^{(p-1)}(x). \tag{2.8}$$

Proof. We have

$$\begin{aligned}
\frac{(k-1)!}{(k-p)!}&\lambda_j(\cdot - x)^{k-p}\\
&= \sum_{r<k} \psi_j^{(r)}(\tau_j)(-1)^r\,[(\partial/\partial s)^{k-1-r}\,(s-x)^{k-p}/(k-p)!]\,|_{s=\tau_j}\\
&= \sum_{r=p-1}^{k-1} \psi_j^{(r)}(\tau_j)(-1)^r\,(\tau_j - x)^{r-p+1}/(r-p+1)!\\
&= \sum_{s=0}^{k-p} \psi_j^{(s+p-1)}(\tau_j)(-1)^{s+p-1}\,(\tau_j - x)^s/s!\\
&= (-1)^{p-1}\sum_{s=0}^{k-p} (\psi_j^{(p-1)})^{(s)}\,(\tau_j)(x-\tau_j)^s/s!\\
&= (-1)^{p-1}\,\psi_j^{(p-1)}(x),
\end{aligned}$$

since $\psi_j^{(p-1)}$ is a polynomial of degree $k-p$.

COROLLARY. *Under the same assumptions,*

$$\lambda_j(\cdot - x)_+^{k-p} = \frac{(k-p)!}{(k-1)!}(-1)^{p-1}\,\psi_j^{(p-1)}(x)(\tau_j - x)_+^0\,. \tag{2.9}$$

Proof. If $\tau_j \leqslant x-$, then both sides of (2.9) are trivially zero; hence, (2.9) holds in this case. If $\tau_j \geqslant x+$, then

$$\lambda_j(\cdot - x)_+^{k-p} = \lambda_j(\cdot - x)^{k-p},$$

and (2.9) follows from the lemma.

THEOREM 2.1. *Let λ_j be given by* (2.5) *and* (2.6). *If*

$$x_j + \leqslant \tau_j \leqslant x_{j+k} - \tag{2.10}$$

then

$$\lambda_j N_{i,k} = \delta_{ij}, \qquad all\ i.$$

Proof. By the corollary to Lemma 2.1,

$$\lambda_j g_k(\cdot; x) = \psi_j^+(x), \tag{2.11}$$

with

$$\psi_j^+(x) = \psi_j(x)(\tau_j - x)_+^0 .$$

Further,

$$g_k(s; x) = (s - x)^{k-1} + (-1)^k g_k(x; s);$$

hence, from (2.1) and (2.11),

$$\begin{aligned} \lambda_j N_{i,k} &= (x_{i+k} - x_i)(-1)^k \lambda_j g_k(\cdot; x_i, \ldots, x_{i+k}) \\ &= (x_{i+k} - x_i)(-1)^k \psi_j^+(x_i, \ldots, x_{i+k}). \end{aligned} \tag{2.12}$$

Since ψ_j vanishes at $x_{j+1}, \ldots, x_{j+k-1}$, and ψ_j^+ vanishes for $x > \tau_j$, it follows, with (2.10), that

$$\psi_j^+(x_r) = 0, \qquad \text{for all} \quad r > j.$$

Therefore, for $i > j$,

$$\psi_j^+(x_i, \ldots, x_{i+k}) = 0,$$

or, with (2.12),

$$\lambda_j N_{i,k} = 0, \qquad \text{for} \quad i > j.$$

If $i < j$, then ψ_j^+ agrees with ψ_j at $x_i, \ldots, x_{i+k}$; since ψ_j is a polynomial of degree $< k$, we. therefore, have

$$\psi_j^+(x_i, \ldots, x_{i+k}) = \psi_j(x_i, \ldots, x_{i+k}) = 0,$$

or, with (2.12),

$$\lambda_j N_{i,k} = 0, \qquad \text{for} \quad i < j.$$

Finally, if $i = j$, then ψ_j^+ agrees with the kth degree polynomial

$$p(x) = \psi_j(x)(x_{j+k} - x)/(x_{j+k} - x_j)$$

at $x_i, \ldots, x_{i+k}$; therefore,

$$\psi_j^+(x_i, \ldots, x_{i+k}) = p^{(k)}(x)/k! = (-1)^k/(x_{j+k} - x_j)$$

or with (2.12)

$$\lambda_j N_{i,k} = 1, \qquad \text{for} \quad i = j,$$

proving the theorem.

Remark. The fact that the set of normalized B-splines (2.1) is a basis for $S_\pi{}^k$ is a rather easy consequence of this theorem, as is the statement that the set

$$N_{i,k}, \ldots, N_{i+r,k},$$

is linearly independent, considered as a set of functions on

$$[x_{i+k-1}, x_{i+r+1}].$$

COROLLARY 1. *If for all j, τ_j satisfies* (2.10), *then F_π as defined by* (2.4)–(2.6) *is a linear projector with range $S_\pi{}^k$.*

Proof. F_π is, off hand, a linear map on $C_\pi^{(k-1)}$ with range in $S_\pi{}^k$. Hence, it suffices to show that

$$F_\pi f = f, \qquad \text{for all} \quad f \in S_\pi{}^k. \tag{2.13}$$

But, if $f \in S_\pi{}^k$, then by [6]

$$f = \sum_i a_i(f)\, N_{i,k},$$

for certain coefficients $a_i(f)$. Since each λ_j is a *local* linear functional, it follows that

$$\lambda_j f = \lambda_j \left(\sum_i a_i(f)\, N_{i,k} \right) = \sum_i a_i(f)(\lambda_j N_{i,k}).$$

Hence, by Theorem 2.1,

$$\lambda_j f = a_j(f), \qquad \text{all } j, \tag{2.14}$$

proving (2.13).

We mention a few obvious but noteworthy consequences of the preceding results.

Since

$$(\cdot - x)^{k-p} \in S_\pi{}^k, \qquad \text{for} \quad 1 \leqslant p \leqslant k,$$

we get, with (2.8), from the preceding corollary that

$$(s - x)^{k-p} = \frac{(k-p)!}{(k-1)!} \sum_j (-1)^{p-1}\, \psi_j^{(p-1)}(x)\, N_{j,k}(s), \tag{2.15}$$

which is Marsden's identity [9] in one of its many equivalent forms. But since (2.8) holds for arbitrary τ_j, we then get from (2.15) the following.

COROLLARY 2. *If F_π is given by* (2.4)–(2.6), *then*

$$F_\pi p = p, \qquad \text{for all polynomials } p \text{ of degree} < k. \tag{2.16}$$

Again, since $g_k(\cdot; x_i) \in S_\pi{}^k$ for any i, Corollary 1 to Theorem 2.1 and (2.9) imply that

$$(s - x_i)_+^{k-1} = \sum_j \psi_j{}^+(x_i)\, N_{j,k}(s), \tag{2.17}$$

which can, of course, also be derived directly from (2.15).

Note further that (2.14) offers a convenient way to calculate the coordinate vector $(a_j(f))$ for $f \in S_\pi{}^k$ with respect to the normalized B-spline basis, once the numbers $f^{(r)}(\tau_j)$, $r < k$, all j, are known. In practice, we would make use of the fact that the restriction (2.10) allows several of the τ_j's to coincide making it possible to calculate k of the coefficients $a_j(f)$ from the k pieces of data $f^{(r)}(\tau_j)$, $r < k$. Incidentally, if $\tau_j = x_i$ for some i, then only $f^{(r)}(\tau_j)$, $r < k - 1$, are needed for the calculation of $a_j(f) = \lambda_j f$, since then the coefficient $\omega_{j,k-1} = \psi_j(\tau_j)$ of $f^{(k-1)}(\tau_j)$ in (2.5) vanishes.

3. The Quasiinterpolant on a Finite Partition

Since the quasiinterpolant $F_\pi f$ to f provides a *local* approximation, it is readily adapted to the practically important problem of constructing approximations by polynomial splines on a finite partition.

Let k be a positive integer, and let $\pi = \{x_i\}_0^N$ be a k-extended partition for the finite interval $[a, b]$. Specifically,

$$a = x_0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{N-1} < x_N = b, \tag{3.1}$$

with coincidences among the x_j's restricted to no more than k consecutive of them, i.e.,

$$x_j < x_{j+k}, \qquad \text{all } j.$$

If $S_\pi{}^k$ denotes the linear space of all polynomial splines of order k on π, then

$$\{N_{j,k} \mid j = -k + 1, \ldots, N - 1\},$$

is a basis for $S_\pi{}^k$. Here we have augmented π by additional, rather arbitrary points

$$x_{-k+1} \leqslant \cdots \leqslant x_{-1} \leqslant a, \qquad b \leqslant x_{N+1} \leqslant \cdots \leqslant x_{N+k-1}. \tag{3.2}$$

For $f \in C_\pi^{(k-1)}$, we define $F_\pi f$, as before, by

$$F_\pi f = \sum_{j=-k+1}^{N-1} (\lambda_j f) \, N_{j,k} \,, \tag{3.3}$$

with λ_j given by (2.5)–(2.6) and τ_j satisfying (2.10) as well as the additional restriction that

$$\tau_j \in [a, b], \qquad \text{all } j. \tag{3.4}$$

Note that this additional restriction is compatible with (2.10).

Typically, we might choose

$$\tau_j = \begin{cases} a+, & j + k/2 < 0, \\ x_{j+k/2}, & 0 \leqslant j + k/2 \leqslant N, \\ b-, & N < j + k/2, \end{cases} \tag{3.5}$$

where

$$x_{i+1/2} = (x_i + x_{i+1})/2.$$

With this choice, we get, for $0 \leqslant j + k/2 \leqslant N$,

$$\lambda_j f = \begin{cases} f(x_{j+1/2}), & k = 1, \\ f(x_{j+1}), & k = 2, \\ f(x_{j+3/2}) - \frac{1}{8}(\Delta x_{j+1})^2 f^{(2)}(x_{j+3/2}), & k = 3, \\ f(x_{j+2}) + \frac{1}{3}\Delta^2 x_{j+1} f^{(1)}(x_{j+2}) - \frac{1}{6}\Delta x_{j+1}\, \Delta x_{j+2} f^{(2)}(x_{j+2}), & k = 4, \end{cases}$$

to give some explicit formulas.

It readily follows from Theorem 2.1 that the map F_π is a linear projector with range S_π^k. In preparation for later sections, we now consider the error

$$e = f - F_\pi f.$$

For simplicity, we assume that π is a *strict* partition, i.e.,

$$x_i < x_{i+1}\,, \qquad \text{all } i.$$

We also assume that $f \in C^{(k-1)}[a, b]$. For $x \in [a, b]$, let

$$(T_x f)(s) = \sum_{r<k} f^{(r)}(x)(s - x)^r/r!$$

and

$$R_x = f - T_x f.$$

Then, $T_x f$ is a polynomial of degree $< k$; hence, as F_π reproduces such polynomials (see Corollary 2 to Theorem 2.1) we get

$$F_\pi(T_x f) = T_x f.$$

On the other hand,

$$(T_x f)^{(r)}(x) = f^{(r)}(x), \qquad \text{for all} \quad r < k;$$

therefore,

$$e^{(r)}(x) = f^{(r)}(x) - (F_\pi f)^{(r)}(x) = -(F_\pi R_x)^{(r)}(x), \qquad r < k,$$

or

$$e^{(r)}(x) = -\sum_j (\lambda_j R_x)\, N_{j,k}^{(r)}(x), \qquad r < k. \tag{3.6}$$

Hence, estimating $e^{(r)}(x)$ amounts to bounding the k terms

$$|(\lambda_j R_x)\, N_{j,k}^{(r)}(x)|,$$

for which $N_{j,k}(x)$ is nonzero. This can be done as follows.

Since $R_x^{(\rho)}(x) = 0$ for $\rho < k$, expansion of $R_x^{(r)}$ in a partial Taylor series around x gives

$$\begin{aligned} R_x^{(r)}(y) &= R_x^{(k-1)}(\xi)(y-x)^{k-1-r}/(k-1-r)! \\ &= (f^{(k-1)}(\xi) - f^{(k-1)}(x))(y-x)^{k-1-r}/(k-1-r)!, \end{aligned}$$

for some $\xi = \xi(y, r)$ between y and x. With this, the definition (2.5)–(2.6) of λ_j implies

$$\lambda_j R_x = \sum_{\rho<k} \omega_{\rho,j}[f^{(k-1)}(\xi_\rho) - f^{(k-1)}(x)](\tau_j - x)^{k-1-\rho}/(k-1-\rho)!.$$

Therefore, with

$$A_{\rho,j}^r(x) = |\,\omega_{k-1-\rho,j}(\tau_j - x)^\rho\, N_{j,k}^{(r)}(x)|/\rho!, \tag{3.7}$$

we get

$$|(\lambda_j R_x)\, N_{j,k}^{(r)}(x)| \leqslant \omega(f^{(k-1)}, |\,\tau_j - x\,|) \sum_{\rho<k} A_{\rho,j}^r(x), \tag{3.8}$$

where $\omega(g, \cdot)$ denotes the modulus of continuity of g.

It remains to find suitable bounds for the quantities $A_{\rho,j}^r(x)$. With (2.6), we have, more explicitly,

$$A_{\rho,j}^r(x) = \frac{1}{\rho!}\,|\,\psi_j^{(\rho)}(\tau_j)(\tau_j - x)^\rho\, N_{j,k}^{(r)}(x)|/(k-1)!,$$

where

$$\psi_j(x) = (x_{j+1} - x) \cdots (x_{j+k-1} - x).$$

Since

$$0 \leqslant N_{j,k}(x) \leqslant 1,$$

$$x_j \leqslant \tau_j, x \leqslant x_{j+k},$$

this gives at once

$$A^0_{\rho,j}(x) \leqslant K_\rho(\Delta_j)^{k-1}, \tag{3.9}$$

where K_ρ is some constant depending only on ρ and where

$$\Delta_j = \max_{j \leqslant i < j+k} (x_{i+1} - x_i). \tag{3.10}$$

For $r > 0$, a more careful analysis is required to establish in which cases $A^r_{\rho,j}(x)$ can be bounded independently of the local mesh ratio.

First, using the facts that

$$N^{(r+1)}_{i,m}(x) = (m-1)\left\{\frac{N^{(r)}_{i,m-1}(x)}{x_{i+m-1} - x_i} - \frac{N^{(r)}_{i+1,m-1}(x)}{x_{i+m} - x_{i+1}}\right\}$$

and that $0 \leqslant N_{i,m}(x) \leqslant 1$, one can prove by induction the following.

LEMMA 3.1. *For $r < k$, there exist a constant $C = C(k, r)$ and an integer $n = n(k, r, j, x)$ such that*

$$|N^{(r)}_{j,k}(x)| \leqslant C/(x_{n+k-r} - x_n)^r,$$

with

$$x_j \leqslant x_n \leqslant x \leqslant x_{n+k-r} \leqslant x_{j+k}. \tag{3.11}$$

Further,

$$\psi^{(\rho)}_j(s)/\rho! = \pm \sum_{I_\rho} \prod_{\sigma \in I_\rho} (\sigma - s),$$

where the sum is taken over all subsets I_ρ of $x_{j+1}, ..., x_{j+k-1}$ of cardinality $k - 1 - \rho$. Hence, with $C_1 = C_1(k, r, \rho)$ some constant,

$$A^r_{\rho,j}(x) \leqslant C_1 \frac{|\tau_j - x|^\rho}{(x_{n+k-r} - x_n)^r} \sum_{I_\rho} \prod_{\sigma \in I_\rho} |\sigma - \tau_j|. \tag{3.12}$$

Since both τ_j and x are in $[x_j, x_{j+k}]$, this gives

$$A^r_{\rho,j}(x) \leqslant C^r_{\rho,j}(\Delta_j)^{k-1-r}, \tag{3.13}$$

where the constant $C^r_{\rho,j}$ may depend on the local mesh ratio.

LEMMA 3.2. *If* $2r \leqslant k$, *and*

$$x_n \leqslant \tau_j \leqslant x_{n+k-r}, \tag{3.14}$$

then the constant $C^r_{\rho,j}$ *in* (3.13) *can be chosen independently of* τ_j.

Proof. By (3.12), the task of bounding $A^r_{\rho,j}(x)$ amounts to bounding $\rho!$ terms of the form

$$\left(\prod_{\sigma\in J} (\tau_j - \sigma)\right) \Big/ (x_{n+k-r} - x_n)^r, \tag{3.15}$$

where

$$J = I_\rho \cup \overbrace{\{x, \ldots, x\}}^{\rho \text{ times}}.$$

We claim that at least $k - r$ elements of J lie between x_n and x_{n+k-r}. Indeed, since at least $k - r$ of $\{x_{j+1}, \ldots, x_{j+k-1}\}$ lie in $[x_n, x_{n+k-r}]$, and I_ρ contains $k - 1 - \rho$ of the $k - 1$ points $\{x_{j+1}, \ldots, x_{j+k-1}\}$, it follows that

$$\#\{x_i \in I_\rho \mid x_n \leqslant x_i \leqslant x_{n+k-r}\} \geqslant k - 1 - \rho + k - r - (k - 1) = k - r - \rho,$$

where, by (3.11), the ρ x's all lie in $[x_n, x_{n+k-r}]$. It follows that at least $k - r$ of the $k - 1$ factors,

$$|\tau_j - \sigma|, \qquad \sigma \in J,$$

are less than or equal to $x_{n+k-r} - x_n$. Therefore, if $r \leqslant k - r$, then all terms in the denominator of (3.15) can be cancelled against suitable terms in the numerator without increasing the value of the expression, which proves the lemma.

It remains to discuss the condition (3.14) which should be satisfied if we are to get bounds (for $2r \leqslant k$) which do not depend on π. Since Lemma 3.1 gives no information about n beyond the condition (3.11), we must choose τ_j so as to satisfy (3.14) for all n satisfying (3.11). Hence, with

$$m = [k/2],$$

we need to pick τ_j so that

$$x_n \leqslant \tau_j \leqslant x_{n+k-m},$$

for all n such that

$$x_j \leqslant x_n \leqslant x \leqslant x_{n+k-m} \leqslant x_{j+k},$$

if we want (3.14) to hold for all $r \leqslant k/2$. The choice,

$$x_{j+m} \leqslant \tau_j \leqslant x_{j+k-m},$$

will accomplish this for all $x \in [x_j, x_{j+k}]$. But note that, for certain j, $[x_j, x_{j+k}]$ will not lie entirely in $[a, b] = [x_0, x_N]$. As we are only concerned with $x \in [x_0, x_N]$, it is, therefore, sufficient to choose each τ_j subject to

$$x_{\min(j+k,N)-k+m} \leqslant \tau_j \leqslant x_{\max(j,0)+k-m} \tag{3.16}$$

to have (3.11) hold for all $x \in [a, b]$. Specifically, the choice (3.5) satisfies (3.16).

The preceding discussion proves the following.

THEOREM 2.1. *If* $f \in C^{(k-1)}[a, b]$, *and* F_π *is given by* (3.1), (2.5), (2.6), *and* (3.16), *then*

$$\| f^{(r)} - (F_\pi f)^{(r)} \|_\infty \leqslant K_r \omega(f^{(k-1)}, |\pi|) \, |\pi|^{k-1-r}, \qquad r < k,$$

where, for $r \leqslant k/2$, K_r *is independent of* π (or f), *while, for* $r > k/2$, K_r *depends on the local mesh ratio*

$$M_\pi = \max_{|i-j|=1} (x_{i+1} - x_i)/(x_{j+1} - x_j).$$

Here

$$|\pi| = \max_{0 < i \leqslant N} (x_i - x_{i-1}),$$

$$\| g \|_\infty = \max_{a \leqslant x \leqslant b} | g(x) |.$$

4. THE MULTIVARIATE QUASIINTERPOLANT

In this section we extend the quasiinterpolant construction to include functions of n variables. We use boldface type to denote points in R^n,

$$\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$$

with $\mathbf{x}_i$ the ith component of $\mathbf{x}$. For each $\nu = 1,..., n$, let

$$\pi_\nu = \{\mathbf{x}_{l,\nu}\}_{l \in Z}$$

be a k-extended partition of R and set $\pi = \pi_1 \times \cdots \times \pi_n$. Thus, as $\mathbf{j} = (\mathbf{j}_1, ..., \mathbf{j}_n)$ varies over Z^n, π can be characterized as the set of nodes $\mathbf{x}_\mathbf{j}$ with

$$\mathbf{x}_\mathbf{j} = (\mathbf{x}_{\mathbf{j}_1,1}, ..., \mathbf{x}_{\mathbf{j}_n,n}).$$

For $\mathbf{j} \in Z^n$ and $\mathbf{x} \in R^n$ we let

$$N_{\mathbf{j},k}(\mathbf{x}) = N_{\mathbf{j}_1,k}(\mathbf{x}_1) \cdots N_{\mathbf{j}_n,k}(\mathbf{x}_n), \tag{4.1}$$

the B-spline of degree $k - 1$ (in each variable) which has the n-dimensional interval or cube

$$(\mathbf{x}_{\mathbf{j}}, \mathbf{x}_{\mathbf{j}+k\mathbf{1}})$$

as its support. Here we have used the abbreviation

$$\mathbf{1} = (1,\ldots, 1).$$

Correspondingly, we define the linear functional $\tilde{\lambda}_{\mathbf{j}}$ by the rule

$$\tilde{\lambda}_{\mathbf{j}} f = \sum_{0 \leq \boldsymbol{\alpha}_\nu < k} \omega_{\mathbf{j},\boldsymbol{\alpha}} (D^{\boldsymbol{\alpha}} f)(\boldsymbol{\tau}_{\mathbf{j}}), \tag{4.2}$$

where $\boldsymbol{\tau}_{\mathbf{j}}$ is some point in the support of $N_{\mathbf{j},k}$ and (see (2.5), (2.6))

$$\begin{aligned} \omega_{\mathbf{j},\boldsymbol{\alpha}} &= \prod_{\nu=1}^{n} \omega_{\mathbf{j},\boldsymbol{\alpha}_\nu,\nu}, \\ \omega_{\mathbf{j},\boldsymbol{\alpha}_\nu,\nu} &= (-1)^{k-1-\boldsymbol{\alpha}_\nu} D^{k-1-\boldsymbol{\alpha}_\nu} \psi_{\mathbf{j}_\nu,\nu}(\boldsymbol{\tau}_{\mathbf{j},\nu})/(k-1)!, \\ \psi_{\mu,\nu}(t) &= \prod_{i=1}^{k-1} (\mathbf{x}_{\mu+i,\nu} - t). \end{aligned} \tag{4.3}$$

Hence, $\tilde{\lambda}_{\mathbf{j}}$ is (an extension of) the tensor product $\bigotimes_{\nu=1}^{n} \tilde{\lambda}_{\mathbf{j},\nu}$ with

$$\tilde{\lambda}_{\mathbf{j},\nu} g = \sum_{\alpha < k} \omega_{\mathbf{j},\alpha,\nu} (D^{\alpha} g)(\boldsymbol{\tau}_{\mathbf{j},\nu}).$$

Therefore, by Theorem 2.1,

$$\tilde{\lambda}_{\mathbf{i}} N_{\mathbf{j},k} = \delta_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \text{if } \mathbf{i} = \mathbf{j} \\ 0 & \text{if } \mathbf{i} \neq \mathbf{j}. \end{cases}$$

This shows that the rule

$$\tilde{F}_\pi f = \sum_{\mathbf{j} \in Z^n} (\tilde{\lambda}_{\mathbf{j}} f)\, N_{\mathbf{j},k} \tag{4.4}$$

(with the sum taken pointwise as in the univariate case) defines a linear projector $\tilde{F}_\pi$ on the linear space of functions f in R^n which have $k - 1$ continuous derivatives in each variable.[2] The map $\tilde{F}_\pi$ enjoys all the properties possessed by one-dimensional quasiinterpolants; it is projective into its range $S_\pi{}^k$; it is local; it approximates to the correct order. However, it has the

[2] We shall denote this class of functions by C^{k-1}, or $C^{k-1}(\Omega)$ if the dependence on the region Ω is important.

disadvantage of being defined only on $W_\infty^{n(k-1)}$, while in practice it is important to work with the larger Sobolev space W_∞^{k-1}.[3]

By simply omitting in (4.2) all terms $(D^\alpha f)(\tau_j)$ with $|\alpha| \geq k$, we obtain the map

$$F_\pi f = \sum_j (\lambda_j f)\, N_{j,k} \tag{4.5a}$$

with

$$\lambda_j f = \sum_{|\alpha| < k} \omega_{j,\alpha} (D^\alpha f)(\tau_j). \tag{4.5b}$$

F_π maps all of $W_\infty^{k-1} \cap C^{k-1}$ into $S_\pi{}^k$ but fails to be a linear projector, although it is local and, more importantly, $F_\pi f$ approximates f to the correct order. This latter point will be verified systematically in Section 5; however, it seems appropriate to give an intuitive justification here. The key lies in the nature of the multivariate Taylor expansion

$$T_x f(\mathbf{y}) = \sum_{|\alpha| < k} (D^\alpha f)(\mathbf{x})(\mathbf{y} - \mathbf{x})^\alpha / \alpha!.$$

The error estimate in Section 3 centered around the ability of F_π to reproduce this Taylor polynomial which, in turn, requires that F_π reproduce the monomials

$$\mathbf{y}^\alpha, \qquad |\alpha| < k.$$

Since $S_\pi{}^k$ contains the larger set of monomials

$$\mathbf{y}^\alpha, \qquad \alpha_\nu < k, \qquad \text{all } \nu,$$

and $\tilde{F}_\pi$ is a linear projector with $S_\pi{}^k$ as its range, it follows that $\tilde{F}_\pi$ reproduces $\mathbf{y}^\alpha$ for $|\alpha| < k$. But, whenever $f(\mathbf{y}) = \mathbf{y}^\alpha$ with $|\alpha| < k$, then $D^\beta f = 0$ for $|\beta| \geq k$. Hence, since λ_j is obtained from $\tilde{\lambda}_j$ by omitting all terms $D^\beta f$ with $|\beta| \geq k$, then

$$\tilde{\lambda}_j f = \lambda_j f.$$

Therefore, finally, $F_\pi f = \tilde{F}_\pi f = f$. This proves the following lemma.

LEMMA 4.1. *With $\tilde{F}_\pi$ and F_π defined by* (4.4) *and* (4.5), *respectively*,

$$\tilde{F}_\pi p = F_\pi p = p,$$

for all polynomials p of total degree $< k$.

[3] We shall denote by $W_\infty{}^m$ the class of functions f whose derivatives up to order m are bounded; i.e., $\sup_{\mathbf{x}} |D^\alpha f(\mathbf{x})| < \infty$ for $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$.

If f is defined only in a region $\Omega \subset R^n$, then it is possible to define $F_\pi f$ using only the values of f and its derivatives in Ω. Clearly this will be the case if τ_j is chosen to be any point in

$$\Omega \cap (x_j, x_{j+k1}).$$

However, further conditions are needed on the choice of the evaluation points τ_j to prevent unnecessary dependence of the error estimates on mesh ratios (see Section 3). Accordingly, we shall assume in the sequel that the τ_j's have been centered in the support of $N_{j,k}$, except near $\partial\Omega$, in the following manner. With

$$k' = [k/2],$$

put

$$\tau_j = \mathbf{x}_{j'} \tag{4.6}$$

with $\mathbf{j}'$ chosen so that

$$|\mathbf{j} + k'\mathbf{1} - \mathbf{j}'| \leqslant |\mathbf{j} + k'\mathbf{1} - \mathbf{m}| \qquad \text{for all } \mathbf{m} \quad \text{with} \quad \mathbf{x}_\mathbf{m} \in \bar{\Omega}. \tag{4.7}$$

5. Error Estimates

In this section we shall work with a partition π of R^n consisting of distinct nodes $\mathbf{x}_j$, $\mathbf{j} \in Z^n$. Having established error estimates for such partitions and having determined when these estimates depend on mesh ratios, we can then let nodal points coalesce to obtain the k-extended partition setting.

Let Ω be a region in R^n. For each $f \in W_\infty^{k-1}(\Omega) \cap C^{k-1}(\Omega)$, we define $F_\pi f$ by (4.5a) and (4.5b) with the evaluation points τ_j being chosen according to (4.6) and (4.7). Our first result is a local estimate of the error in the interval

$$C_j = (\mathbf{x}_j, \mathbf{x}_{j+1}) \tag{5.1}$$

in terms of values of f in

$$C_{j,k} = (x_{j-(k+1)1}, x_{j+k1}). \tag{5.2}$$

THEOREM 5.1. *Let* $(\mathbf{x}_j, x_{j+1}) \cap \Omega \neq \varnothing$, *and let* $f \in C^{k-1}(\Omega \cap C_{j,k})$. *Then for* $|\mathbf{s}| < k$

$$\max_{\mathbf{x} \in C_j} |D^\mathbf{s} e(\mathbf{x})| \leqslant K_\mathbf{s} \Delta_j^{k-1-|\mathbf{s}|} \omega_{k-1}(f; \Delta_j; C_{j,k}), \tag{5.3}$$

where

$$\Delta_j = \max_{1 \leqslant \nu \leqslant n} \Delta_{j_\nu}, \qquad \Delta_{j_\nu} = |x_{j+k1_\nu} - x_{j-(k-1)1,\nu}| \tag{5.4}$$

and

$$\omega_{k-1}(f; \Delta_{\mathbf{j}}; C_{\mathbf{j},k}) = \max_{|\gamma|=k-1} \omega(D^\gamma f; \Delta_{\mathbf{j}}; C_{\mathbf{j},k}). \tag{5.5}$$

The constants $K_{\mathbf{s}}$ *are independent of* f *and* $\mathbf{j}$, *and are also independent of the partition* π *for*

$$0 \leqslant \mathbf{s}_\nu \leqslant [k/2], \qquad 1 \leqslant \nu \leqslant n.$$

For the other values of $\mathbf{s}$ *they depend only on the mesh ratios in* $C_{\mathbf{j},k}$.

Proof. For $\mathbf{x} \in C_{\mathbf{j}}$, let

$$(T_{\mathbf{x}}f)(\mathbf{y}) = \sum_{|\boldsymbol{\alpha}|<k} (D^{\boldsymbol{\alpha}}f)(\mathbf{x})(\mathbf{y}-\mathbf{x})^{\boldsymbol{\alpha}}/\boldsymbol{\alpha}! \tag{5.6}$$

and

$$R_{\mathbf{x}}f = f - T_{\mathbf{x}}f. \tag{5.7}$$

From Lemma 4.1 we see that F_π reproduces the Taylor polynomial $T_{\mathbf{x}}f$; i.e.,

$$(F_\pi T_{\mathbf{x}}f)(\mathbf{y}) = (T_{\mathbf{x}}f)(\mathbf{y}). \tag{5.8}$$

Therefore, the application of the differential operator $D^{\mathbf{s}}$, $|\mathbf{s}| < k$, to the spline $F_\pi R_{\mathbf{x}}f$ (for fixed $\mathbf{x}$) gives

$$D^{\mathbf{s}}F_\pi R_{\mathbf{x}}f = D^{\mathbf{s}}F_\pi f - D^{\mathbf{s}}F_\pi T_{\mathbf{x}}f = D^{\mathbf{s}}F_\pi f - D^{\mathbf{s}}T_{\mathbf{x}}f. \tag{5.9}$$

But

$$(D^{\mathbf{s}}T_{\mathbf{x}}f)(\mathbf{x}) = D^{\mathbf{s}}f(\mathbf{x}); \tag{5.10}$$

hence, we have the following representation of the error:

$$(D^{\mathbf{s}}e)(\mathbf{x}) = (D^{\mathbf{s}}f)(\mathbf{x}) - (D^{\mathbf{s}}F_\pi f)(\mathbf{x}) = -(D^{\mathbf{s}}F_\pi R_{\mathbf{x}}f)(\mathbf{x}), \qquad \mathbf{x} \in C_{\mathbf{j}}. \tag{5.11}$$

Using (4.5b) we rewrite (5.11) as

$$\begin{aligned}(D^{\mathbf{s}}e)(\mathbf{x}) &= -\sum_{\boldsymbol{l}} \lambda_{\mathbf{j}+\boldsymbol{l}}(R_{\mathbf{x}}f)\, D^{\mathbf{s}}N_{\mathbf{j}+\boldsymbol{l},k}(\mathbf{x}) \\ &= -\sum_{\boldsymbol{l}} \sum_{|\boldsymbol{\alpha}|<k} \omega_{\mathbf{j}+\boldsymbol{l},\boldsymbol{\alpha}}(D^{\boldsymbol{\alpha}}R_{\mathbf{x}}f)(\boldsymbol{\tau}_{\mathbf{j}+\boldsymbol{l}})\, D^{\mathbf{s}}N_{\mathbf{j}+\boldsymbol{l},k}(\mathbf{x}),\end{aligned} \tag{5.12}$$

where $\mathbf{x} \in C_{\mathbf{j}}$ and the sum is over all $\boldsymbol{l}$ satisfying

$$-k+1 \leqslant \boldsymbol{l}_\nu \leqslant 0, \qquad 1 \leqslant \nu \leqslant n. \tag{5.13}$$

Expanding $(D^{\boldsymbol{\alpha}}R_{\mathbf{x}}f)(\boldsymbol{\tau}_{\mathbf{j}+\boldsymbol{l}})$ in a Taylor series about $\mathbf{x} \in C_{\mathbf{j}}$ and using

$$(D^{\boldsymbol{\gamma}}R_{\mathbf{x}}f)(\mathbf{x}) = 0, \qquad |\boldsymbol{\gamma}| < k,$$

we obtain

$$\lambda_{\mathbf{j}+l}(R_{\mathbf{x}}f) = \sum_{|\boldsymbol{\alpha}|<k} \sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|=k-1} \omega_{\mathbf{j}+l,\boldsymbol{\alpha}}(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}R_{\mathbf{x}}f)(\boldsymbol{\xi})(\mathbf{x} - \boldsymbol{\tau}_{\mathbf{j}+l})^{\boldsymbol{\beta}}/\boldsymbol{\beta}!, \quad (5.14)$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{j}, \boldsymbol{l})$ is a point on the line joining $\mathbf{x}$ and $\boldsymbol{\tau}_{\mathbf{j}}$. But for $|\boldsymbol{\alpha} + \boldsymbol{\beta}| = k - 1$,

$$|(D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}R_{\mathbf{x}}f)(\boldsymbol{\xi})| = |D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}f(\mathbf{x}) - D^{\boldsymbol{\alpha}+\boldsymbol{\beta}}f(\boldsymbol{\xi})| \leqslant \omega_{k-1}(f; \Delta_{\mathbf{j}}; C_{\mathbf{j},k}). \quad (5.15)$$

Therefore, to complete the error estimates it suffices to show that there is a constant $K_{\mathbf{s}}$ such that

$$|\omega_{\mathbf{m},\boldsymbol{\alpha}}(\mathbf{x} - \tau_{\mathbf{m}})^{\boldsymbol{\beta}} D^{\mathbf{s}}N_{\mathbf{m},k}(\mathbf{x})| \leqslant K_{\mathbf{s}}\Delta_{\mathbf{j}}^{k-1-|\mathbf{s}|}, \quad (5.16)$$

for $|\boldsymbol{\alpha} + \boldsymbol{\beta}| = k - 1$, $\mathbf{x} \in C_{\mathbf{j}}$, and

$$\mathbf{j}_\nu - k + 1 \leqslant \mathbf{m}_\nu \leqslant \mathbf{j}_\nu, \qquad 1 \leqslant \nu \leqslant n. \quad (5.17)$$

Note that (5.16) is clearly true for some constant $K_{\mathbf{s}}$ which depends only on the mesh ratios in $C_{\mathbf{j},k}$. We shall now determine those values of $\mathbf{s}$ where $K_{\mathbf{s}}$ is actually independent of the mesh ratios. We first note that (5.16) is a product

$$\prod_{\nu=1}^{n} |\omega_{\mathbf{m},\boldsymbol{\alpha}_\nu,\nu}[\mathbf{x}_\nu - \boldsymbol{\tau}_{\mathbf{m},\nu}]^{k-1-\boldsymbol{\alpha}_\nu} \frac{\partial^{\mathbf{s}_\nu}}{\partial \mathbf{x}_\nu{}^{\mathbf{s}_\nu}} N_{\mathbf{m}_\nu,k}(\mathbf{x}_\nu)|. \quad (5.18)$$

Moreover, the analysis given in Section 3, in particular Lemmas 3.1 and 3.2, applies to each term in the product (5.18). Observe that $\tau_{\mathbf{m},\nu}$, $1 \leqslant \nu \leqslant n$, satisfies (3.16) if $\boldsymbol{\tau}_{\mathbf{m}}$ is chosen by the rule (4.6) and (4.7). We conclude that each term in (5.18) is bounded by

$$K_{\mathbf{s}_\nu}\Delta_{\mathbf{j}_\nu}^{k-1-\mathbf{s}_\nu},$$

for $1 \leqslant \nu \leqslant n$, where $K_{\mathbf{s}_\nu}$ is independent of the mesh ratios for $0 \leqslant \mathbf{s}_\nu \leqslant [k/2]$; consequently (5.16) holds with

$$K_{\mathbf{s}} = \prod_{\nu=1}^{n} K_{\mathbf{s}_\nu},$$

and $K_{\mathbf{s}}$ is independent of the mesh ratios for $0 \leqslant \mathbf{s}_\nu \leqslant [k/2]$, $1 \leqslant \nu \leqslant n$.

Since the constant $K_{\mathbf{s}}$ is independent of $\mathbf{j}$, the global error estimate

$$\| D^{\mathbf{s}}e \|_{L_\infty(\Omega)} \leqslant K_{\mathbf{s}} |\pi|^{k-1-|\mathbf{s}|} \omega_{k-1}(f; |\pi|; \Omega_\pi), \quad (5.19)$$

where

$$\Omega_\pi = \bigcup_{\mathbf{j}} C_{\mathbf{j},k}$$

and the union is over all $\mathbf{j}$ such that $C_\mathbf{j} \cap \Omega \neq \varnothing$ follows immediately from (5.3). Sharper estimates can be obtained if we assume that Ω is locally convex in the sense that each $\mathbf{x} \in C_{\mathbf{j},k}$ can be "seen" by the evaluation points $\boldsymbol{\tau}_{\mathbf{j}+\boldsymbol{l}}$ in the following sense.

DEFINITION 5.1. The region Ω is said to be locally convex with respect to the partition π and the evaluation points $\{\boldsymbol{\tau}_\mathbf{j}\}$ if for each $\mathbf{j}$, $C_\mathbf{j} \cap \Omega \neq \varnothing$, and for each $\boldsymbol{l}$ satisfying (5.13), the line from $\boldsymbol{\tau}_{\mathbf{j}+\boldsymbol{l}}$ to any $\mathbf{x} \in C_\mathbf{j}$ lies in Ω.

We note that the class of locally convex regions in the sense of Definition 5.1 includes not only all convex regions but also nonconvex regions such as rectangular polygons whose sides are parallel to the coordinate axes. However, regions which have reentrant corners whose interior angles exceed $3\pi/2$ are not locally convex; see, *e.g.*, Fig. 1.

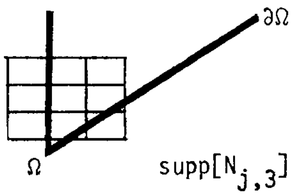

FIGURE 1

If Ω is locally convex, then the point $\boldsymbol{\xi}$ in (5.14) lies in Ω. Hence (5.15) can be rewritten

$$|(D^{\alpha+\beta} R_\mathbf{x} f)(\boldsymbol{\xi})| \leq \omega_{k-1}(f; \Delta_\mathbf{j}; C_{\mathbf{j},k} \cap \Omega),$$

and we conclude that (5.19) holds with Ω replacing Ω_π. Moreover, we need only assume $f \in C^{k-1}(\Omega)$ for this estimate.

Remark 1. The previous error estimates remain valid if the terms

$$D^\alpha f(\tau_\mathbf{j}), \qquad |\boldsymbol{\alpha}| < k,$$

appearing in $F_\pi f$ are replaced with difference quotients with accuracy $\mathcal{O}(|\pi|^{k-|\alpha|})$. Unfortunately this substitution, while of great importance in practice, will in general introduce mesh ratio dependence in the error $D^\mathbf{s} e$ for *all* $|\mathbf{s}| < k$. It remains an open question as to whether a discretization can be found such that there is no mesh ratio dependence for say $\mathbf{s} = \mathbf{0}$.

Remark 2. If the function f is not in C^{k-1}, a modified quasiinterpolant can still be constructed. Specifically, let $f \in C^q[\Omega]$ for $0 \leq q \leq k-1$, and put

$$F_{\pi,q}f = \sum_{\mathbf{j}} \sum_{|\alpha| \leq q} \omega_{\mathbf{j},\alpha} D^\alpha f(\boldsymbol{\tau}_{\mathbf{j}})\, N_{\mathbf{j},k}.$$

Then the estimate (5.3) holds if $k-1$ is replaced with q.

In applications [4, 13] it is important to construct spline approximations to functions f for which we only know that f lies in the Sobolev space $W_r^k(\Omega)$; i.e.,

$$\|f\|_{k,r,\Omega} = \left\{ \int_\Omega \sum_{|\alpha| \leq k} |D^\alpha f(\mathbf{x})|^r \, d\mathbf{x} \right\}^{1/r} < \infty. \tag{5.20}$$

The major problem with the quasiinterpolant $F_\pi f$ is that it may not even be defined if the number of spatial dimensions n is too large; for example, $f = \ln \ln(\mathbf{x}_1^2 + \mathbf{x}_2^2)$ satisfies (5.20) with $k = 1$ and $r = 2$, yet this function is not bounded in any region containing the origin. We shall show, however, that if the quasiinterpolant $F_\pi f$ is defined, then estimates similar to (5.19) are valid in integral norms like (5.20).

To fix ideas let us suppose that the map F_π contains q derivatives. Thus, for a strict partition $q = k - 1$, and $q = k - 2$ if the evaluation points are centered. For the Hermite approximation, on the other hand, k is an even integer and $q = [k]/2 - 1$. This is obtained from a strict partition by letting $[k]/2 - 1$ successive nodes coalesce in each variable. To test whether F_π is defined on functions f satisfying (5.20) we use the *Sobolev embedding theorem* [15] which states that

$$\sup_{|\beta| \leq q} \sup_{x \in \Omega} |D^\beta f(\mathbf{x})| < \infty,$$

provided

$$k > m/r + q. \tag{5.21}$$

Let us assume (5.21) holds, and let us study the error $f - F_\pi f$ in the cube $C_{\mathbf{j},k}$, which we rewrite as C to simplify notation. Our analysis is somewhat similar to the foregoing estimates, except that we no longer use the Taylor polynomial $T_{\mathbf{x}} f$ associated with f. This requires more information about f than is provided by (5.20). To get an alternate polynomial which is close to f in C, we use the *change of scale trick* now familiar in finite element theory [13, 14]. In particular, consider the affine transformation of C onto the unit cube $Q = (\mathbf{0}, \mathbf{1})$. This is a dilatation with the ratio of volumes being $\mathcal{O}(h^n)$, where h is the diameter of C. Now, it is known [15] that if

$$|F|_{k,r,Q} = \max_{|\beta| = k} \|D^\beta F\|_{k,r,Q} < \infty, \tag{5.22}$$

then there is a polynomial P of total degree at most $k - 1$ such that

$$\| F - P \|_{k,r,Q} \leqslant K \| F \|_{k,r,Q} \tag{5.23}$$

holds for some absolute constant $K > 0$. In fact, (5.23) states that (5.22) is equivalent to the natural norm on the quotient space $W_r^{k-1}(Q)/P_{k-1}$ (P_{k-1} denoting the space of polynomials of total degree at most $k - 1$). Let F be the image of f under the dilatation $C \to Q$ and let the polynomial p be the inverse image of P. Then after a change of scale (5.23) becomes

$$\| f - p \|_{l,r,C} \leqslant K_l h^{k-l} \| f \|_{k,r,C}, \qquad 0 \leqslant l \leqslant q, \tag{5.24}$$

where the constant K_l depends on the mesh ratios. In fact, we shall be rather crude and not keep track of the latter. Defining the residual by

$$R(\mathbf{x}) = f(\mathbf{x}) - p(\mathbf{x}),$$

we have

$$F_\pi f - f = F_\pi R - R$$

since $F_\pi p = p$. Thus,

$$\| F_\pi f - f \|_{l,r,C} \leqslant \| F_\pi R \|_{l,r,C} + \| R \|_{l,r,C}. \tag{5.25}$$

The second term on the right of (5.25) is exactly (5.24) and, hence, is of order $\mathcal{O}(h^{k-l})$. To estimate the first term we recall

$$D^{\mathbf{s}} F_\pi R(\mathbf{x}) = \sum_{\mathbf{l}} \lambda_{\mathbf{j}+\mathbf{l}}(R)\, D^{\mathbf{s}} N_{\mathbf{j}+\mathbf{l}}(\mathbf{x}).$$

Now, by assumption, $\lambda_{\mathbf{j}+\mathbf{l}}$ involves only derivatives up to order q with the αth derivative multiplied by a weight of order $h^{|\alpha|}$ (see (4.2) and (4.3)). Thus, with (5.24) we conclude that

$$| \lambda_{\mathbf{j}+\mathbf{l}}(R) | \leqslant K h^k \| f \|_{k,r,C}.$$

Since $D^{\mathbf{s}} N_{\mathbf{j}+\mathbf{l}}$ is of order $h^{k-|\mathbf{s}|}$, we have

$$\| F_\pi R \|_{l,r,C} \leqslant K h^{k-l} \| f \|_{k,r,C}.$$

Summing over all $C = C_{j,k}$ gives the following.

THEOREM 5.2. *Let* (5.20) *and* (5.21) *hold. Then*

$$\| f - F_\pi f \|_{l,r,\Omega} \leqslant K_l \, |\pi|^{k-l} \, \| f \|_{k,r,\Omega}, \qquad 0 \leqslant l \leqslant q, \tag{5.26}$$

where K_l is an absolute constant depending at most on the mesh ratios.

6. Numerical Results

In this section we shall illustrate the accuracy of approximation by quasi-interpolants with some specific examples. For simplicity we shall confine attention to the cubic case ($k = 4$) with uniform partitions.

As a first example we consider the cubic spline approximation of

$$f(x) = \exp(x), \tag{6.1}$$

in $0 \leqslant x \leqslant 1$. Partition $[0, 1]$ into intervals with mesh length h, and let

$$x_j = (j - 4)h, \qquad 1 \leqslant j \leqslant N + 7, \tag{6.2}$$

where $N = 1/h$. The quasiinterpolant can be written

$$F_\pi f(x) = \sum_{j=1}^{N+3} \sum_{r=0}^{3} \omega_{j,r} f^{(r)}(\tau_j)\, N_{j,4}(x) \tag{6.3}$$

(see (2.4)–(2.6)). If the evaluation point τ_j is centered in the support of $N_{j,4}$,

$$\tau_j = x_{j+2}, \tag{6.4}$$

we have (see (3.5)ff.)

$$\omega_{j,r} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r = 1 \\ -h^2/6 & \text{if } r = 2 \\ 0 & \text{if } r = 3 \end{cases}. \tag{6.5}$$

For $j = 1$ and $j = N + 3$, $\tau_j \notin [0, 1]$; hence, for these cases we modify the foregoing by taking

$$\tau_1 = x_4 = 0, \qquad \tau_{N+3} = x_{N+4} = 1. \tag{6.6}$$

This gives

$$\omega_{1,r} = \begin{cases} 1 & \text{if } r = 0 \\ -h & \text{if } r = 1 \\ h^2/3 & \text{if } r = 2 \\ 0 & \text{if } r = 3 \end{cases}, \tag{6.7}$$

$$\omega_{N+3,r} = \begin{cases} 1 & \text{if } r = 0 \\ h/3 & \text{if } r = 1 \\ -h^2/3 & \text{if } r = 2 \\ 0 & \text{if } r = 3 \end{cases}. \tag{6.8}$$

Note that for $h \leqslant x \leqslant 1 - h$,

$$F_\pi f(x) = \sum \{f(x_{j+2}) - (h^2/6) f^{(2)}(x_{j+2})\}\, N_{j,4}(x). \tag{6.9}$$

As indicated in Section 5, the derivatives in (6.9) can be replaced with a difference quotient having accuracy $\mathcal{O}(h^2)$; in particular this gives

$$F_{\pi,1}f(x) = \sum_j \{[8f(x_{j+2}) - f(x_{j+3}) - f(x_{j-1})]/6\} N_{j,4}(x), \qquad x_5 \leqslant x \leqslant x_{N+3}, \tag{6.10}$$

with suitable modifications for x in $[0, h]$ and $[1 - h, 1]$. For simplicity we shall use (6.10) to define $F_{\pi,1}f$ everywhere in $0 \leqslant x \leqslant 1$, thus explicitly using values of f outside $[0, 1]$ to compute values of $F_{\pi,1}f$.

In Table 6.1 we shall compare the accuracies of (6.3) and (6.10) with the accuracy of the *spline interpolant*

$$F_{\pi,2}f(x) = \sum_j C_j N_{j,4}(x), \tag{6.11}$$

TABLE 6.1

Errors in the Cubic Spline Quasiinterpolant, Discretized Quasiinterpolant, and Interpolant of (6.13)
Notation: $1.2 - n$ means 1.2×10^{-n}

	Function values		
h	$e^{(0)}_{0,h}$	$e^{(0)}_{1,h}$	$e^{(0)}_{2,h}$
1/4	0.156–3	0.298–3	0.263–4
1/8	0.103–4	0.190–4	0.169–4
1/16	0.663–6	0.122–5	0.107–6
	Derivatives		
h	$e^{(1)}_{0,h}$	$e^{(1)}_{1,h}$	$e^{(1)}_{2,h}$
1/4	0.416–3	0.538–3	0.321–3
1/8	0.479–4	0.562–4	0.409–4
1/16	0.554–5	0.609–5	0.512–6

which is defined by

$$\begin{aligned} (F_{\pi,2}f)(x_j) &= f(x_j), \qquad 4 \leqslant j \leqslant N+4 \\ (F_{\pi,2}f)'(x_j) &= f'(x_j), \qquad j = 4 \quad \text{and} \quad N+4. \end{aligned} \tag{6.12}$$

In this table we shall let

$$e^{(r)}_{l,h} = \max_{0 \leqslant x \leqslant 1} |f^{(r)}(x) - (F_{\pi,l}f)^{(r)}(x)|, \tag{6.13}$$

for $0 \leqslant l \leqslant 2$, $0 \leqslant r \leqslant 1$ and $F_{\pi,0} = F_\pi$.

As a second example we shall consider the spline bicubic approximation of

$$f(x, y) = \exp(x, y), \tag{6.14}$$

in the L-shaped region shown in Fig. 2.

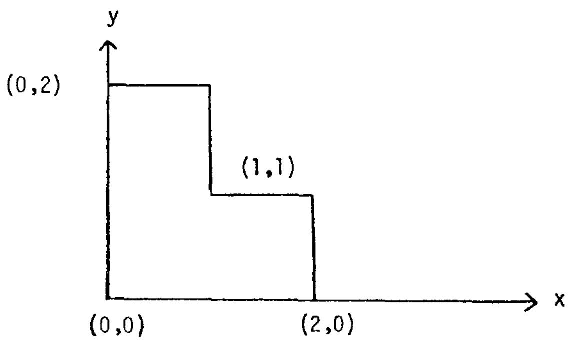

FIG. 2. L-shaped region.

The quasiinterpolant is defined by (4.5), where we take

$$\tau_{\mathbf{j}} = (\mathbf{j}_1 h + 2h, \mathbf{j}_2 h + 2h) = (x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2}),$$

if the latter is in Ω; otherwise, we let $\tau_{\mathbf{j}}$ be the boundary point closest to $(x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2})$. If the point $(x, y) \in \Omega$ is at least a distance h from $\partial\Omega$,

$$F_{\pi,0} f(x, y) = \sum \{f(x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2}) - h^2 \Delta f(x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2})/16\} N_{\mathbf{j},4}(x, y), \tag{6.15}$$

with the discretized version being

$$F_{\pi,1} f(x, y) = \sum \{f(x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2}) - h^2 \Delta_h f(x_{\mathbf{j}_1+2}, y_{\mathbf{j}_2+2})/16\} N_{\mathbf{j},4}(x, y),$$

$$\Delta_h f(x, y) = [f(x + h, y) + f(x - h, y) + f(x, y + h) + f(x, y - h) - 4f(x, y)]/h^2. \tag{6.16}$$

For simplicity we shall define $F_{\pi,1} f$ everywhere in Ω by (6.16) (thus, using values of f outside Ω). As in Table 6.1 we shall let

$$e_{l,h}^{(0)} = \max_{(x,y)\in\Omega} | f(x, y) - F_{\pi,l} f(x, y) |,$$

$$e_{l,h}^{(1)} = \max_{(x,y)\in\Omega} \left| \frac{\partial f}{\partial x}(x, y) - \frac{\partial}{\partial x} F_{\pi,l} f(x, y) \right|.$$

Because of symmetry

$$e_{l,h}^{(1)} = \max_{(x,y)\in\Omega} \left| \frac{\partial f}{\partial y}(x, y) - \frac{\partial}{\partial y} F_{\pi,l} f(x, y) \right|.$$

TABLE 6.2

Errors in the Bicubic Spline Quasiinterpolant and Discretized Quasiinterpolant of (6.15)
Notation: $1.2 - n$ means 1.2×10^{-n}

h	$e^{(0)}_{0,h}$	$e^{(0)}_{1,h}$	$e^{(1)}_{0,h}$	$e^{(1)}_{1,h}$
1/4	0.196–1	0.275–1	0.604–1	0.787–1
1/8	0.115–2	0.159–2	0.402–2	0.481–2
1/16	0.708–4	0.977–4	0.218–3	0.279–3

Appendix

Local Spline Approximation by Moments

Let k be a positive integer and let $\pi = \{x_i\}_0^N$ be a partition of the finite interval $[a, b]$, as in (3.1), augmented by suitable points, as in (3.2). Birkhoff's scheme of "local spline approximation by moments" [3] (as described in [5] for even *and* odd k) approximates $f \in C^{(k-1)}[a, b]$ by

$$(P_\pi f)(x) = \sum_{r<k} f^{(r)}(a)(x - a)^r/r! + \frac{1}{(k-1)!} \int_a^b Q_\pi(x - \cdot)_+^{k-1} \, df^{(k-1)}. \tag{A.1}$$

Here, Q_π is k-point central polynomial interpolation. In this scheme, a function g defined on $[t_{-m}, t_{N+m}]$ is approximated on $[a, b]$ by $Q_\pi g$, where

$$(Q_\pi g)(x) = p_i(x) \qquad \text{for} \quad x \in [x_{i-1+k/2}, x_{i+k/2}),$$

p_i being the polynomial of degree $< k$ which agrees with g at $x_i, \ldots, x_{i+k-1}$, $i = -m, \ldots, N - m$, and

$$m = [k/2].$$

It follows that $Q_\pi g$ can be written in Lagrange form,

$$Q_\pi g = \sum_i g(x_i) \, W_i,$$

where, for each i, W_i is the function on $[a, b]$ determined, e.g., by

$$W_i(x) = Q_\pi \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

With this, (A.1) can be written

$$(P_\pi f)(x) = \sum_{r<k} f^{(r)}(a)(x-a)^r/r! + \frac{1}{(k-1)!} \sum_{i\geqslant -m} (x-x_i)_+^{k-1} \int_a^b W_i \, df^{(k-1)} \tag{A.2}$$

THEOREM A. *For all $f \in C^{(k-1)}[a, b]$,*

$$P_\pi f = F_\pi f, \tag{A.3}$$

where $F_\pi f$ is the quasiinterpolant to f described in Section 3, *with the τ_j's given by* (3.5).

Proof. Since both P_π and F_π reproduce polynomials of degree $< k$, it is sufficient to prove (A.3) under the additional assumption

$$f^{(r)}(a) = 0, \qquad \text{all} \quad r < k. \tag{A.4}$$

For such f,

$$\lambda_j f = \frac{1}{(k-1)!} \sum_{r<k} (-)^r \psi_j^{(r)}(\tau_j) f^{(k-1-r)}(\tau_j) = \frac{1}{(k-1)!} \int_a^{\tau_j} \psi_j \, df^{(k-1)}$$

as we verify by repeated integration by parts. Hence,

$$(k-1)!\, F_\pi f = \sum_j \int_a^{\tau_j} \psi_j \, df^{(k-1)} N_{j,k} \tag{A.5}$$

On the other hand, with (A.2) and (A.4),

$$(k-1)!\, P_\pi f = \sum_i (x-x_i)_+^{k-1} \int_a^b W_i \, df^{(k-1)}$$

By (2.17),

$$(x-x_i)_+^{k-1} = \sum_j \psi_j^+(x_i)\, N_{j,k}(x),$$

where we can take

$$\psi_j^+(x) = (x-x_{j+1})_+ \cdots (x-x_{j+k-1})_+$$

Hence,

$$(k-1)!\, P_\pi f = \sum_i \sum_j \psi_j^+(x_i) \int_a^b W_i \, df^{(k-1)} N_{j,k}$$

$$= \sum_j A_j(f)\, N_{j,k},$$

with

$$A_j(f) = \int_a^b \sum_i \psi_j^+(x_i)\, W_i\, df^{(k-1)}$$
$$= \int_a^b Q_\pi \psi_j^+\, df^{(k-1)}.$$

By (A.5), it, therefore, suffices to show that

$$\int_a^b Q_\pi \psi_j^+\, df^{(k-1)} = \int_a^{\tau_j} \psi_j\, df^{(k-1)}. \tag{A.6}$$

But this can be seen as follows. We have

$$\psi_j^+(x_i) = \begin{cases} \psi_j(x_i), & \text{for} \quad i < j + k \\ 0, & \text{for} \quad i > j. \end{cases}$$

Therefore, if p_i is the polynomial of degree $< k$ which agrees with ψ_j^+ at $x_i, \ldots, x_{i+k-1}$, then

$$p_i = \begin{cases} \psi_j, & \text{for} \quad i \leq j \\ 0, & \text{for} \quad i > j. \end{cases}$$

But this implies, using the definition of Q_π, that

$$(Q_\pi \psi_j^+)(x) = \psi_j(x)(x_{j+k-2} - x)_+^0.$$

which, with the Definition (3.5) of τ_j, gives (A.6). Q.E.D.

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